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Recurrences and explicit formulae for the expansion and connection coefficients in series of Bessel polynomials

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Abstract

A formula expressing explicitly the derivatives of Bessel polynomials of any degree and for any order in terms of the Bessel polynomials themselves is proved. Another explicit formula, which expresses the Bessel expansion coefficients of a general-order derivative of an infinitely differentiable function in terms of its original Bessel coefficients, is also given. A formula for the Bessel coefficients of the moments of one single Bessel polynomial of certain degree is proved. A formula for the Bessel coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its Bessel coefficients is also obtained. Application of these formulae for solving ordinary differential equations with varying coefficients, by reducing them to recurrence relations in the expansion coefficients of the solution, is explained. An algebraic symbolic approach (using Mathematica) in order to build and solve recursively for the connection coefficients between Bessel–Bessel polynomials is described. An explicit formula for these coefficients between Jacobi and Bessel polynomials is given, of which the ultraspherical polynomial and its consequences are important special cases. Two analytical formulae for the connection coefficients between Laguerre–Bessel and Hermite–Bessel are also developed.

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1. Introduction

Techniques for finding approximate solutions for differential equations, based on classical orthogonal polynomials, are popularly known as spectral methods. Approximating functions in spectral methods are related to polynomial solutions of eigenvalue problems in ordinary differential equations, known as Sturm–Liouville problems. In the past few decades, there has been growing interest in this subject. As a matter of fact, spectral methods provide a competitive alternative to other standard approximation techniques, for a large variety of problems. Initial applications were concerned with the investigation of periodic solutions of boundary value problems using trigonometric polynomials. Subsequently, the analysis was extended to algebraic polynomials. Expansions in orthogonal basis functions were performed, due to their high accuracy and flexibility in computations. Different basis functions lead to different spectral approximations; for instance, trigonometric polynomials for periodic problems, Chebyshev, Legendre, ultraspherical and Jacobi polynomials for non-periodic problems, Laguerre polynomials for problems on the half line, and Hermite polynomials for problems on the whole line.

Chebyshev, Legendre and ultraspherical polynomials are examples of three classes of singular Sturm–Liouville eigenfunctions that have been used in both the solution of boundary value problems (see, for instance, Ben-Yu (1998), Coutsias *et al* (1996), Doha (1990, 2000, 2002a, 2002b, 2003a), Doha and Al-kholi (2001), Doha and Abd-Elhameed (2002), Doha and Helal (1997), Haidvogel and Zang (1979), Siyyam and Syam (1997)) and in computational fluid dynamics (see Canuto *et al* (1988), Helal (2001), Voigt *et al* (1984)). In most of these applications, formulae relating the expansion coefficients of derivatives appearing in the differential equation with those of the function itself are used.

Formulae for the expansion coefficients of a general-order derivative of an infinitely differentiable function in terms of those of the function are available for expansions in Chebyshev (Karageorghis 1988a), Legendre (Phillips 1988), ultraspherical (Karageorghis and Phillips 1989, 1992, Doha 1991), Jacobi (Doha 2002a), Laguerre (Doha 2003b) and Hermite (Doha 2004a) polynomials.

A more general situation which often arises in the numerical solution of differential equations with polynomial coefficients in spectral and pseudospectral methods is the evaluation of the expansion coefficients of the moments of high-order derivatives of infinitely differentiable functions. A formula for the shifted Chebyshev coefficients of the moments of the general-order derivatives of an infinitely differentiable function is given in Karageorghis (1988b). Corresponding results for Chebyshev polynomials of the first and second kinds, Legendre, ultraspherical, Hermite, Laguerre and Jacobi polynomials are given in Doha (1994), Doha and El-Soubhy (1995), Doha (1998, 2003b, 2004a, 2004b) respectively.

Up to now, and to the best of our knowledge, many formulae corresponding to those mentioned previously are not known and are traceless in the literature for the Bessel expansions. This partially motivates our interest in such polynomials. Another motivation is that the theoretical and numerical analysis of numerous physical and mathematical problems very often requires the expansion of an arbitrary polynomial or the expansion of an arbitrary function with its derivatives and moments into a set of orthogonal polynomials. This is, in particular, true for Bessel polynomials. To be precise, the Bessel polynomials form a set of orthogonal polynomials on the unit circle in the complex plane. They are important in certain problems of mathematical physics; for example, they arise in the study of electrical networks and when the wave equation is considered in spherical coordinates.

The paper is organized as follows. In section 2, we give some relevant properties of Bessel polynomials. In section 3, we prove a theorem which relates the Bessel expansion coefficients of the derivatives of a function in terms of its original expansion coefficients. An explicit expression for the derivatives of Bessel polynomials of any degree and for any order as a linear combination of suitable Bessel polynomials themselves is also deduced. In section 4, we prove a theorem which gives the Bessel coefficients of the moments of one single Bessel polynomial of any degree. Another theorem which expresses the Bessel coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its Bessel coefficients is proved in section 5. In section 6, we give an application of these theorems which provides an algebraic symbolic approach (using Mathematica) in order to build and solve recursively for the connection coefficients between Bessel and different polynomial systems.

2. Some properties of Bessel polynomials

The classical sets of orthogonal polynomials of Jacobi, Laguerre and Hermite satisfy second-order differential equations, and also have the property that their derivatives form orthogonal systems. The Bessel polynomials, a fourth class of orthogonal polynomials with these two properties, were introduced by Krall and Frink (1949) in connection with the solution of the wave equation in spherical coordinates.

They define the generalized Bessel polynomial $y_n(x, a, b)$ to be the polynomial of degree n, and with constant term equal to unity, which satisfies the differential equation

$$x^2 y''(x) + [ax + b]y'(x) - n(n + a - 1)y(x) = 0, \qquad b \neq 0, \quad a \neq 0, -1, -2, \dots, \tag{1}$$

where n is a non-negative integer, provided a is not a negative integer or zero, and b is not zero.

It is easy to see that $y_n(bx, a, b)$ is independent of b. Thus it seems preferable to adopt the notation (Al-Salam 1957)

$$Y_n^{(\alpha)}(x) = y_n(x, \alpha + 2, 2),$$

so that $Y_n^{(0)}(x) = y_n(x)$, the ordinary Bessel polynomial. $Y_n^{(\alpha)}(x)$ satisfy the differential equation

$$x^2 y''(x) + [(\alpha + 2)x + 2]y'(x) - n(n + \alpha + 1)y(x) = 0, \qquad \alpha \neq -2, -3, \dots. \tag{2}$$

These polynomials are orthogonal on the unit circle with respect to the weight function

$$\rho^\alpha(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 2)}{\Gamma(k + \alpha + 1)} \left(\frac{-2}{x}\right)^k,$$

satisfying the orthogonality relation

$$\frac{1}{4\pi \mathrm{i}} \int_C Y_n^{(\alpha)}(z) Y_m^{(\alpha)}(z) \rho^\alpha(z)\, \mathrm{d}z = \frac{(-1)^{n+1} n! \Gamma(\alpha + 2)}{(2n + \alpha + 1)\Gamma(n + \alpha + 1)} \delta_{nm},$$

where the integration is around the unit circle surrounding the zero point. $Y_n^{(\alpha)}(x)$ may be generated by using the Rodrigues formula

$$Y_n^{(\alpha)}(x) = 2^{-n} x^{-\alpha}\, \mathrm{e}^{2/x}\, \mathrm{D}^n [x^{2n+\alpha}\, \mathrm{e}^{-2/x}],$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}x$, and explicitly by the formula

$$Y_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n + \alpha + 1)k \left(\frac{x}{2}\right)^k, \tag{3}$$

(for more detail see, for instance, Chihara (1978) and Sánchez-Ruiz and Dehesa (1998)).

Several other authors have contributed to the study of Bessel polynomials, among them are Agarwal (1954), Al-Salam (1957), Carlitz (1957), Evans *et al* (1993), Grosswald (1978), Han and Kwon (1991), and Luke (1969, vol 2).

The following two recurrence relations (which may be found in Koepf and Schmersau (1998)) are of fundamental importance in developing the present work. These are

$$2(n+\alpha+1)(2n+\alpha)Y_{n+1}^{(\alpha)}(x) = (2n+\alpha+1)[(2n+\alpha)(2n+\alpha+2)x+2\alpha]Y_n^{(\alpha)}(x) \\ + 2n(2n+\alpha+2)Y_{n-1}^{(\alpha)}(x), \qquad n \geqslant 1, \tag{4}$$

$$Y_n^{(\alpha)}(x) = \frac{2(n+\alpha+1)}{(2n+\alpha+1)(2n+\alpha+2)(n+1)}\,\mathrm{D}Y_{n+1}^{(\alpha)}(x) + \frac{4}{(2n+\alpha+2)(2n+\alpha)}\,\mathrm{D}Y_n^{(\alpha)}(x) \\ + \frac{2n}{(n+\alpha)(2n+\alpha)(2n+\alpha+1)}\,\mathrm{D}Y_{n-1}^{(\alpha)}(x). \tag{5}$$

Note that the recurrence relation (4) may be used to generate the Bessel polynomials starting from $Y_0^{(\alpha)}(x) = 1$ and $Y_1^{(\alpha)}(x) = \frac{\alpha+2}{2}x + 1$.

Theorem 1. *Let $f(x)$ be a function regular (i.e. analytic) in $|x-a| \leqslant R$, where $R > 0$ and a is any point of the plane. Then $f(x)$ can be expanded in a series of generalized Bessel polynomials of the form $f(x) \sim \sum c_n Y_n^{(\alpha)}(x-a)$, where*

$$c_n = \frac{2^n}{n!}(2n+\alpha+1)\Gamma(n+\alpha+1)\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{\nu!\Gamma(2n+\nu+\alpha+2)}f^{(n+\nu)}(a),$$

and the series is convergent uniformly in $|x-a| \leqslant R$.

Proof. We first suppose that $f(x)$ is regular in $|x| \leqslant R$ and prove that $f(x)$ can be expanded in a series $\sum_{n=0}^{\infty}\gamma_n Y_n^{(\alpha)}(x)$ where

$$\gamma_n = \frac{2^n}{n!}(2n+\alpha+1)\Gamma(n+\alpha+1)\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{\nu!\Gamma(2n+\nu+\alpha+2)}f^{(n+\nu)}(0), \tag{6}$$

and that the series is uniformly convergent in $|x| \leqslant R$. Theorem 1 follows readily when $x-a$ is written for x.

We have (see Sánchez-Ruiz and Dehesa (1998), equation (2.32))

$$x^n = \sum_{i=0}^{n}\pi_{ni}Y_i^{(\alpha)}(x), \qquad n \geqslant 0, \tag{7}$$

where

$$\pi_{ni} = \binom{n}{i}\frac{(-1)^{n-i}2^n(2i+\alpha+1)\Gamma(i+\alpha+1)}{\Gamma(n+i+\alpha+2)}. \tag{8}$$

We now substitute for x^n from (7) in the Taylor expansion $\sum_{n=0}^{\infty}(f^{(n)}(0)/n!)x^n$ of $f(x)$ about the origin to get formally the series $\sum_{n=0}^{\infty}\gamma_n Y_n^{(\alpha)}(x)$, where

$$\gamma_n = \sum_{\nu=0}^{\infty}\pi_{n+\nu,n}f^{(n+\nu)}(0)/(n+\nu)!.$$

Hence inserting the value of $\pi_{n+\nu,n}$ from (8), equation (6) follows at once.

In order to prove that the series $\sum_{n=0}^{\infty}\gamma_n Y_n^{(\alpha)}(x)$ is convergent in $|x| \leqslant R$ we form the sum (see Whittaker (1949), chapters II and III)

$$\omega_n(R) = \sum_i |\pi_{ni}|M_i(R),$$

where $M_i(R) \equiv \max_{|x|=R} \left|Y_i^{(\alpha)}(x)\right| = \sum_{k=0}^{i} 2^{-k}\binom{i}{k}|(i+\alpha+1)_k|R^k$. Applying (8) we obtain after some reduction

$$\omega_n(R) = 2^n \sum_{k=0}^{n} \binom{n}{k} (R/2)^k \sum_{j=0}^{n-k} \binom{n-k}{j} \left| \frac{(2j+2k+\alpha+1)\Gamma(2k+j+\alpha+1)}{\Gamma(n+k+j+\alpha+2)} \right|$$
$$< |2n+\alpha+1||\Gamma(\alpha+1)|R^n \sum_{k=0}^{n} \binom{n}{k} (4/R)^k/|\Gamma(k+\alpha+1)|$$
$$< |2n+\alpha+1|R^n B_n, \tag{9}$$

where $B_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(4/R')^k}{k!}$, $R' = R\mu(\alpha), 0 < \mu(\alpha) < \min_{m\in Z^+} \frac{1}{m}|\alpha+m|$. Effecting the transformation $y = x(1+x)^{-1}$ on the function $x\exp(4x/R') = \sum_{n=0}^{\infty}(4/R')^n x^{n+1}/n!$ it follows that

$$F(y) \equiv (y/(1-y))\exp\{4y/R'(1-y)\} = \sum_{n=0}^{\infty} B_n y^{n+1}.$$

This function is regular in $|y| < 1$; hence by Cauchy's inequality we have

$$B_n < K/\beta^{n+1}, \qquad 0 < \beta < 1,$$

where $K = \max_{|y|=\beta}|F(y)| < \infty$. Inserting this in (9) and making n tend to infinity we obtain

$$\lambda(R) \equiv \lim_{n\to\infty} \sup\{\omega_n(R)\}^{1/n} \leqslant R/\beta,$$

and since β can be taken as near 1 as we please we conclude that $\lambda(R) = R$. According to Cannon (1937) (see also Whittaker (1949), p 11), we infer that the series $\sum_{n=0}^{\infty} \gamma_n Y_n^{(\alpha)}(x)$ is uniformly convergent in $|x| \leqslant R$, as required. □

Remark 1. It is to be noted that the theorem of Nassif (1954, p 408) can be obtained directly from our theorem by taking $\alpha = 0$.

Suppose now we are given a regular function $f(x)$ which is formally expanded in an infinite series of Bessel polynomials,

$$f(x) = \sum_{n=0}^{\infty} a_n Y_n^{(\alpha)}(x), \tag{10}$$

and for the qth derivatives of $f(x)$,

$$\mathrm{D}^q f(x) = \sum_{n=0}^{\infty} a_n^{(q)} Y_n^{(\alpha)}(x), \qquad a_n^{(0)} = a_n, \tag{11}$$

it is possible to derive a recurrence relation involving the Bessel coefficients of successive derivative of $f(x)$. Let us write

$$\mathrm{D}\left[\sum_{n=0}^{\infty} a_n^{(q-1)} Y_n^{(\alpha)}(x)\right] = \sum_{n=0}^{\infty} a_n^{(q)} Y_n^{(\alpha)}(x),$$

then using identity (5) leads to the recurrence relation

$$a_n^{(q-1)} = \frac{2(n+\alpha)}{n(2n+\alpha-1)(2n+\alpha)} a_{n-1}^{(q)} + \frac{4}{(2n+\alpha+2)(2n+\alpha)} a_n^{(q)}$$
$$+ \frac{2(n+1)}{(n+\alpha+1)(2n+\alpha+2)(2n+\alpha+3)} a_{n+1}^{(q)}, \qquad q \geqslant 1, \quad n \geqslant 1. \tag{12}$$

For computing purpose, we see that this equation is not easy to use, since the coefficients on the right-hand side are functions of n. No obvious direct way is available for solving this equation, therefore we resort to the following alternative method that enables one to express $a_n^{(q)}$ in terms of the original expansion coefficients a_k, $k = 0, 1, \ldots$.

3. The derivatives of $Y_n^{(\alpha)}(x)$ and the relation between the coefficients $a_n^{(q)}$ and a_n

The main result of this section is to prove the following theorem which expresses explicitly the Bessel expansion coefficients, $a_n^{(q)}$, of a general-order derivative of an infinitely differentiable function in terms of its original Bessel coefficients, a_n.

Theorem 2. *Suppose that a function $f(x)$ and its qth derivative are formally expanded as in (10) and (11), then*

$$a_n^{(q)} = 2^{-q} \sum_{i=0}^{\infty} (n+i+1)_q (n+q+i+\alpha+1)_q M_n(\alpha+2q, \alpha, n+i) a_{n+q+i}, \qquad n \geqslant 0, \quad q \geqslant 1, \tag{13}$$

where

$$M_i(\alpha, \beta, n) = (-1)^n (2i+\beta+1) \frac{(\alpha-\beta)_{n-i} (-n)_i (\alpha+n+1)_i}{i!(\beta+i+1)_{n+1}}. \tag{14}$$

The following two lemmas are needed to proceed with the proof of the theorem.

Lemma 1 (Sánchez-Ruiz and Dehesa 1998)**.** *The connection problem between Bessel polynomials with different parameters is*

$$Y_n^{(\alpha)}(x) = \sum_{i=0}^{n} M_i(\alpha, \beta, n) Y_i^{(\beta)}(x),$$

where the connection coefficients $M_i(\alpha, \beta, n)$ are given as in (14).

Lemma 2. *The derivatives of Bessel polynomials of any degree in terms of Bessel polynomials with the same parameter are given by*

$$\mathrm{D}^q Y_n^{(\alpha)}(x) = 2^{-q} (n-q+1)_q (n+\alpha+1)_q \sum_{i=0}^{n-q} M_i(\alpha+2q, \alpha, n-q) Y_i^{(\alpha)}(x). \tag{15}$$

Proof. Al-Salam (1957) has proved that

$$\mathrm{D} Y_n^{(\alpha)}(x) = \tfrac{1}{2} n(n+\alpha+1) Y_{n-1}^{(\alpha+2)}(x), \qquad n \geqslant 1, \tag{16}$$

and therefore

$$\mathrm{D}^q Y_n^{(\alpha)}(x) = 2^{-q} (n-q+1)_q (n+\alpha+1)_q Y_{n-q}^{(\alpha+2q)}(x). \tag{17}$$

From lemma 1 and identity (17), we obtain (15). □

Proof of theorem 2. Now, on differentiating (10) q times and making use of (15), we find

$$\begin{aligned} \mathrm{D}^q f(x) &= \sum_{n=q}^{\infty} a_n D^q Y_n^{(\alpha)}(x) \\ &= 2^{-q} \sum_{n=q}^{\infty} a_n (n-q+1)_q (n+\alpha+1)_q \sum_{i=0}^{n-q} M_i(\alpha+2q, \alpha, n-q) Y_i^{(\alpha)}(x), \end{aligned} \tag{18}$$

expanding (18) and collecting similar terms, we obtain (13) which completes the proof of theorem 2. □

Remark 2. It is to be noted here that the formula for $a_n^{(q)}$ given by (13) is the exact solution of the difference equation (12), and it also worth noting that based on theorem 1, one can show that the series (13) is convergent.

4. Bessel coefficients of the moments of one single Bessel polynomial of any degree

For the evaluation of Bessel coefficients of the moments of higher order derivatives of an infinitely differentiable function, the following theorem is needed.

Theorem 3

$$x^m Y_j^{(\alpha)}(x) = \sum_{n=0}^{2m} a_{mn}(j) Y_{j+m-n}^{(\alpha)}(x), \qquad m \geqslant 0, \quad j \geqslant 0, \tag{19}$$

where

$$a_{mn}(j) = \frac{(-1)^{j-n} 2^m m! j! (2j+2m-2n+\alpha+1)\Gamma(j+m-n+\alpha+1)}{(j+m-n)!(2m-n)!\Gamma(j+\alpha+1)\Gamma(2j+2m-n+\alpha+2)} \times \sum_{k=\max(0,j-n)}^{\min(j+m-n,j)} \binom{j+m-n}{k} \frac{(-1)^k \Gamma(j+k+\alpha+1)\Gamma(j+2m-n-k+1)}{(j-k)!(n+k-j)!}. \tag{20}$$

Proof. We use the induction principle to prove this theorem. In view of recurrence relation

$$x Y_j^{(\alpha)}(x) = \frac{2(j+\alpha+1)}{(2j+\alpha+1)(2j+\alpha+2)} Y_{j+1}^{(\alpha)}(x) - \frac{2\alpha}{(2j+\alpha)(2j+\alpha+2)} Y_j^{(\alpha)}(x) - \frac{2j}{(2j+\alpha)(2j+\alpha+1)} Y_{j-1}^{(\alpha)}(x), \qquad j \geqslant 0,$$

we may write

$$x Y_j^{(\alpha)}(x) = a_{10}(j) Y_{j+1}^{(\alpha)}(x) + a_{11}(j) Y_j^{(\alpha)}(x) + a_{12}(j) Y_{j-1}^{(\alpha)}(x), \tag{21}$$

and this in turn shows that (19) is true for $m = 1$. Proceeding by induction, assuming that (19) is valid for m, we want to prove that

$$x^{m+1} Y_j^{(\alpha)}(x) = \sum_{n=0}^{2m+2} a_{m+1,n}(j) Y_{j+m-n+1}^{(\alpha)}(x). \tag{22}$$

From (21) and assuming the validity for m, we have

$$x^{m+1} Y_j^{(\alpha)}(x) = \sum_{n=0}^{2m} a_{mn}(j) \big[a_{10}(j+m-n) Y_{j+m-n+1}^{(\alpha)}(x) + a_{11}(j+m-n) Y_{j+m-n}^{(\alpha)}(x) + a_{12}(j+m-n) Y_{j+m-n-1}^{(\alpha)}(x) \big].$$

Collecting similar terms, we get

$$x^{m+1} Y_j^{(\alpha)}(x) = a_{m0}(j) a_{10}(j+m) Y_{j+m+1}^{(\alpha)}(x) + [a_{m1}(j) a_{10}(j+m-1) + a_{m0}(j) a_{11}(j+m)] Y_{j+m}^{(\alpha)}(x) + \sum_{n=2}^{2m} [a_{mn}(j) a_{10}(j+m-n)$$

$$+a_{m,n-1}(j)a_{11}(j+m-n+1)+a_{m,n-2}(j)a_{12}(j+m-n+2)]Y^{(\alpha)}_{j+m-n+1}(x)$$
$$+[a_{m,2m}(j)a_{11}(j-m)+a_{m,2m-1}(j)a_{12}(j-m+1)]Y^{(\alpha)}_{j-m}(x)$$
$$+a_{m,2m}(j)a_{12}(j-m)Y^{(\alpha)}_{j-m-1}(x). \tag{23}$$

It can be easily shown that

$$\begin{aligned}
a_{m+1,0}(j) &= a_{m0}(j)a_{10}(j+m),\\
a_{m+1,1}(j) &= a_{m1}(j)a_{10}(j+m-1)+a_{m0}(j)a_{11}(j+m),\\
a_{m+1,n}(j) &= a_{mn}(j)a_{10}(j+m-n)+a_{m,n-1}(j)a_{11}(j+m-n+1)\\
&\quad +a_{m,n-2}(j)a_{12}(j+m-n+2),\\
a_{m+1,2m+1}(j) &= a_{m,2m}(j)a_{11}(j-m)+a_{m,2m-1}(j)a_{12}(j-m+1),\\
a_{m+1,2m+2}(j) &= a_{m,2m}(j)a_{12}(j-m),
\end{aligned}$$

and accordingly, formula (23) becomes

$$x^{m+1}Y^{(\alpha)}_j(x)=\sum_{n=0}^{2m+2}a_{m+1,n}(j)Y^{(\alpha)}_{j+m-n+1}(x),$$

which completes the induction and proves the theorem. □

Corollary 1. *It can be easily shown that the expansion coefficients $a_{mn}(j)$ of theorem 3 satisfy the recurrence relation*

$$a_{mn}(j)=\sum_{k=0}^{2}a_{m-1,n+k-2}(j)a_{1,2-k}(j+m-n-k+1), \qquad n=0,1,\ldots,2m, \tag{24}$$

where

$$a_{1k}(j)=\begin{cases}\dfrac{2(j+\alpha+1)}{(2j+\alpha+1)(2j+\alpha+2)}, & k=0,\\ -\dfrac{2\alpha}{(2j+\alpha)(2j+\alpha+2)}, & k=1, \qquad a_{00}(j)=1\\ -\dfrac{2j}{(2j+\alpha)(2j+\alpha+1)}, & k=2,\end{cases} \tag{25}$$

with

$$a_{m-1,-\ell}(j)=0, \quad \forall \ell>0, \qquad a_{m-1,r}(j)=0, \quad r=2m-1,2m.$$

Corollary 2. *One can show that*

$$x^mY^{(\alpha)}_j(x)=\sum_{n=0}^{j+m}a_{m,j+m-n}(j)Y^{(\alpha)}_n(x), \qquad j\geqslant 0, \quad m\geqslant 0, \tag{26}$$

and

$$x^m=\sum_{n=0}^{m}a_{m,m-n}(0)Y^{(\alpha)}_n(x), \qquad m\geqslant 0, \tag{27}$$

where

$$a_{m,m-n}(0)=\binom{m}{n}\frac{(-1)^{m-n}2^m(2n+\alpha+1)\Gamma(n+\alpha+1)}{\Gamma(m+n+\alpha+2)}. \tag{28}$$

Formula (27) is in complete agreement with Sánchez-Ruiz and Dehesa (1998) and Zarzo *et al* (1997).

5. Bessel coefficients of the moments of a general-order derivative of an infinitely differentiable function

In this section, we state and prove a theorem which relates the Bessel coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its Bessel coefficients.

Theorem 4. *Assume that $f(x)$, $\mathrm{D}^q f(x)$ and $x^\ell Y_j^{(\alpha)}(x)$ have the Bessel expansions (10), (11) and (19) respectively, and assume also that*

$$x^\ell \left(\sum_{i=0}^{\infty} a_i^{(q)} Y_i^{(\alpha)}(x)\right) = \sum_{i=0}^{\infty} b_i^{q,\ell} Y_i^{(\alpha)}(x) = I^{q,\ell}, \tag{29}$$

then the connection coefficients $b_i^{q,\ell}$ are given by

$$b_i^{q,\ell} = \begin{cases} \displaystyle\sum_{k=0}^{\ell-1} a_{\ell,k+\ell-i}(k) a_k^{(q)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell) a_{k+\ell}^{(q)}, & 0 \leqslant i \leqslant \ell, \\ \displaystyle\sum_{k=i-\ell}^{\ell-1} a_{\ell,k+\ell-i}(k) a_k^{(q)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell) a_{k+\ell}^{(q)}, & \ell+1 \leqslant i \leqslant 2\ell-1, \\ \displaystyle\sum_{k=i-2\ell}^{i} a_{\ell,k+2\ell-i}(k+\ell) a_{k+\ell}^{(q)}, & i \geqslant 2\ell, \end{cases} \tag{30}$$

where the coefficients $a_{mn}(k)$ are as defined in (20).

Proof. Equations (11), (19) and (29) give

$$I^{q,\ell} = \sum_{k=0}^{\infty} a_k^{(q)} \sum_{j=0}^{2\ell} a_{\ell,j}(k) Y_{k+\ell-j}^{(\alpha)}(x). \tag{31}$$

By letting $i = k + \ell - j$, then (31) may be written in the form

$$\begin{aligned} I^{q,\ell} &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x) + \sum_{k=\ell}^{\infty} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x) \\ &= \sum_1 + \sum_2, \end{aligned} \tag{32}$$

where

$$\sum_1 = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x),$$

$$\sum_2 = \sum_{k=\ell}^{\infty} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x).$$

Considering $\sum_1$ first,

$$\begin{aligned} \sum_1 &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{-1} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x) + \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x) \\ &= \sum_{11} + \sum_{12}. \end{aligned} \tag{33}$$

Clearly,

$$\sum_{11} = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{-1} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x) = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=1}^{\ell-k} a_{\ell,k+\ell+i}(k) Y_{-i}^{(\alpha)}(x),$$

hence

$$\sum_{11} = 0. \tag{34}$$

Now,

$$\begin{aligned}\sum_{12} &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x)\\ &= \sum_{i=0}^{\ell}\sum_{k=0}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x) + \sum_{i=\ell+1}^{2\ell-1}\sum_{k=i-\ell}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x),\end{aligned}$$

hence

$$\sum_{12} = \sum_{i=0}^{2\ell-1} \sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x). \tag{35}$$

Substitution of (34) and (35) into (33) yields

$$\sum_{1} = \sum_{i=0}^{2\ell-1} \sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) Y_i^{(\alpha)}(x). \tag{36}$$

If when considering $\sum_2$, one takes $k+\ell$ instead of k, then it is not difficult to show that

$$\sum_{2} = \sum_{i=0}^{\infty} \sum_{k=\max(0,i-2\ell)}^{i} a_{k+\ell}^{(q)} a_{\ell,k+2\ell-i}(k+\ell) Y_i^{(\alpha)}(x). \tag{37}$$

Substitution of (36) and (37) into (32) gives the required results of (30) and completes the proof of theorem 4. □

6. Recurrence relations for connection coefficients between Bessel and different polynomial systems

Let $f(x)$ have the Bessel expansion (10), and assume that it satisfies the linear non-homogeneous differential equation of order n

$$\sum_{i=0}^{n} p_i(x) f^{(i)}(x) = g(x), \tag{38}$$

where $p_0, p_1, \ldots, p_n \neq 0$ are polynomials in x, and the coefficients of Bessel series of the function $g(x)$ are known, then formulae (13), (19) and (30) enable one to construct in view of equation (38) the linear recurrence relation of order r,

$$\sum_{j=0}^{r} \alpha_j(k) a_{k+j} = \beta(k), \qquad k \geqslant 0, \tag{39}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_r$ ($\alpha_0 \neq 0, \alpha_r \neq 0$) are polynomials of the variable k.

In this section, we consider the problem of determining the connection coefficients between different polynomial systems. An interesting question is how to transform the

Fourier coefficients of a given polynomial corresponding to an assigned orthogonal basis, into the coefficients of another basis orthogonal with respect to a different weight function. The aim is to determine the so-called connection coefficients of the expansion of any element of the first basis in terms of the elements of the second basis. Suppose V is a vector space of all polynomials over the real or complex numbers and V_m is the subspace of polynomials of degree less than or equal to m. Suppose $p_0(x), p_1(x), p_2(x), \ldots$ is a sequence of polynomials such that $p_n(x)$ is of exact degree n; let $q_0(x), q_1(x), q_2(x), \ldots$ be another such sequence. Clearly, these sequences form a basis for V. It is also evident that $p_0(x), p_1(x), \ldots, p_m(x)$ and $q_0(x), q_1(x), \ldots, q_m(x)$ give two bases for V_m. While working with finite-dimensional vector spaces, it is often necessary to find the matrix that transforms a basis of a given space to another basis. This means that one is interested in the connection coefficients $a_i(n)$ that satisfy

$$q_n(ax+b) = \sum_{i=0}^{n} a_i(n) p_i(x), \tag{40}$$

where a and b are constants. The choice of $p_n(x)$ and $q_n(ax+b)$ depends on the situation. For example, suppose

$$p_n(x) = x^n, q_n(x) = x(x-1)\cdots(x-n+1) = (-1)^n(-x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)},$$

then the connection coefficients $a_i(n)$ are Stirling numbers of the first kind. If the role of these $p_n(x)$ and $q_n(x)$ are interchanged, then we get Stirling numbers of the second kind. These numbers are useful in some combinatorial polynomials (see Abramowitz and Stegun (1970) pp 824–5).

The connection coefficients between many of the classical orthogonal polynomial systems have been determined by different kinds of methods (see, e.g., Szegö (1985), Rainville (1960) and Andrews *et al* (1999)). The aim of this section is to describe a simple procedure (based on the results of theorem 4) in order to find recurrence relations, sometimes easy to solve, between the coefficients $a_i(n)$ when $p_i(x) = Y_i^{(\alpha)}(x)$ and $q_i(x) = Y_i^{(\beta)}(x)$. This gives an alternative and different way to be compared to the approaches of Askey and Gasper (1971), Ronveaux *et al* (1995, 1996), Area *et al* (1998), Godoy *et al* (1997), Koepf and Schmersau (1998), Lewanowicz (2002), Lewanowicz and Woźny (2001), Lewanowicz *et al* (2000), and Sánchez-Ruiz and Dehesa (1998). A nonrecursive way to approach the problem in the case of classical orthogonal polynomials of a discrete variable can be found in Gasper (1974). Moreover, other authors have considered the problem from a recursive point of view (see Koepf and Schmersau (1998)), or even in the classical discrete and q-analogues (cf, Àlvarez-Nodarse *et al* (1998) and Àlvarez-Nodarse and Ronveaux (1996)). Since the connection coefficients $a_i(n)$ depend on two parameters i and n, the most interesting recurrence relations are those which leave one of the parameters fixed. In the cases when the order of the resulting recurrence relation is 1, it defines a hypergeometric term which can be given explicitly in terms of Pochhammer symbol $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$.

6.1. The Bessel–Bessel connection problem

The link between $Y_n^{(\alpha)}(ax+b)$ and $Y_i^{(\beta)}(x)$ given by (40) can easily be replaced by a linear relation involving only $Y_i^{(\beta)}(x)$ using the Bessel differential equation, namely

$$[(ax+b)^2\mathrm{D}^2 + a[(\alpha+2)(ax+b)+2]\mathrm{D} - a^2 n(n+\alpha+1)]Y_n^{(\alpha)}(ax+b) = 0, \tag{41}$$

by substituting

$$Y_n^{(\alpha)}(ax+b) = \sum_{i=0}^{n} a_i(n) Y_i^{(\beta)}(x), \tag{42}$$

with $a_{n+1}(n) = a_{n+2}(n) = \cdots = 0$, and by virtue of formula (29), equation (41) takes the form

$$a^2 I^{2,2} + 2abI^{2,1} + b^2 I^{2,0} + a^2(\alpha+2)I^{1,1} + [ab(\alpha+2)+2]I^{1,0} - a^2 n(n+\alpha+1)I^{0,0} = 0,$$

or

$$a^2 b_i^{2,2} + 2abb_i^{2,1} + b^2 b_i^{2,0} + a^2(\alpha+2)b_i^{1,1} + [ab(\alpha+2)+2]b_i^{1,0} - a^2 n(n+\alpha+1)b_i^{0,0} = 0.$$

By making use of formulae (20) and (30), we obtain

$$\begin{aligned}
&a^2 n(n+\alpha+1)a_i(n) - \frac{2a^2(\alpha+2)(i+\alpha)}{(2i+\alpha-1)(2i+\alpha)} a_{i-1}^{(1)}(n) \\
&\quad + a\left[-b(\alpha+2) + \frac{2(-4i^2 - 4i(\alpha+1) + (a-1)\alpha(\alpha+2))}{(2i+\alpha)(2i+\alpha+2)}\right] a_i^{(1)}(n) \\
&\quad + \frac{2a^2(i+1)(\alpha+2)}{(2i+\alpha+2)_2} a_{i+1}^{(1)}(n) - \frac{4a^2(i+\alpha-1)_2}{(2i+\alpha-3)_4} a_{i-2}^{(2)}(n) \\
&\quad + \frac{4a(i+\alpha)(2a\alpha - b(2i+\alpha-2)(2i+\alpha+2))}{(2i+\alpha-2)_3(2i+\alpha+2)} a_{i-1}^{(2)}(n) \\
&\quad + \left[-b^2 + \frac{4ab\alpha}{(2i+\alpha)(2i+\alpha+2)} + \frac{4a^2(2i(i+1)+\alpha+2i\alpha-\alpha^2)}{(2i+\alpha-1)_2(2i+\alpha+2)_2}\right] a_i^{(2)}(n) \\
&\quad + \frac{4a(i+1)(2a\alpha - b(2i+\alpha)(2i+\alpha+4))}{(2i+\alpha)(2i+\alpha+2)_3} a_{i+1}^{(2)}(n) \\
&\quad - \frac{4a^2(i+1)(i+2)}{(2i+\alpha+2)_4} a_{i+2}^{(2)}(n) = 0, \qquad i \geqslant 0,
\end{aligned} \tag{43}$$

using formula (13) with (43)—and after some manipulation obtain the following recurrence relation,

$$\begin{aligned}
&\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \\
&\quad i = n-1, n-2, \ldots, 0,
\end{aligned} \tag{44}$$

where

$$\delta_{i0} = \frac{a^2(n-i)(i+n+\alpha+1)(i+\beta+1)_4}{(2i+\beta+1)_4},$$

$$\begin{aligned}
\delta_{i1} = -\frac{a(i+1)(i+\beta+2)_3}{(2i+\beta+3)_2}&\left[1 + \frac{b}{2}(\alpha+2i+2)\right. \\
&\left. - \frac{a(\beta(2i+1) + 2(3i+1) + 4n(n+\alpha+1) + (2-2i+\beta)(\alpha+1))}{(2i+\beta+2)(2i+\beta+6)}\right],
\end{aligned}$$

$$\begin{aligned}
\delta_{i2} = (i+1)_2(\beta+i+3)_2&\left[-\frac{b^2}{4} + \frac{a((\beta-\alpha+2)b-2)}{(\beta+i+4)(\beta+i+6)}\right. \\
&\left. + \frac{a^2(2i^2 + 2(\beta+5)i + (3+\beta)(3\alpha-\beta+2) + 6n(n+\alpha+1))}{(\beta+i+3)_2(\beta+i+6)_2}\right],
\end{aligned}$$

$$\begin{aligned}
\delta_{i3} = -\frac{a(i+1)_3(i+\beta+4)}{(2i+\beta+6)_2}&\left[1 - \frac{b}{2}(2i+2\beta-\alpha+8)\right. \\
&\left. + \frac{a(2(\beta-\alpha+2)i - (4+\beta)(3\alpha-2\beta-4) - 4n(n+\alpha+1))}{(\beta+i+3)_2(\beta+i+6)_2}\right],
\end{aligned}$$

$$\delta_{i4} = \frac{a^2(i+1)_4(i+\beta+n+5)(n-i+\alpha-\beta-4)}{(2i+\beta+6)_4},$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3$, and $a_n(n) = \frac{a^n(n+\alpha+1)_n}{(n+\beta+1)_n}$. The solution of (44) is

$$a_i(n) = \frac{(-a)^i(-n)_i(n+\alpha+1)_i}{(i+\beta+1)_i i!} \sum_{k=0}^{n-i} \frac{a^k(-n+i)_k(n+\alpha+i+1)_k}{k!(2i+\beta+2)_k}$$
$$\times {}_2F_0\left[\begin{matrix}-k, -2i-k-\beta-1\\ -\end{matrix}; -\frac{b}{2a}\right], \quad i = 0, 1, \ldots, n. \tag{45}$$

Corollary 3. *In the connection problem*

$$Y_n^{(\alpha)}(ax) = \sum_{i=0}^{n} a_i(n) Y_i^{(\beta)}(x), \tag{46}$$

the coefficients $a_i(n)$ are given by

$$a_i(n) = \frac{(-a)^i(-n)_i(n+\alpha+1)_i}{(i+\beta+1)_i i!} {}_2F_1\left[\begin{matrix}-n+i, n+\alpha+i+1\\ 2i+\beta+2\end{matrix}; a\right], \qquad i = 0, 1, \ldots, n. \tag{47}$$

In the particular case $a = 1$, and if we use the Chu–Vandemonde formula

$${}_2F_1\left[\begin{matrix}-n, c\\ d\end{matrix}; 1\right] = \frac{(d-c)_n}{(d)_n},$$

we find that the expansion coefficients (47) take the form of $M_i(\alpha, \beta, n)$ given by formula (14).

Remark 3. The two connected problems considered by Godoy *et al* (1997, sections 2.1, 2.2, pp 267–8) can be obtained from our problem as two direct special cases.

6.2. The Jacobi–Bessel connection problem

In this problem, we consider the usual standardization of the Jacobi polynomials

$$P_n^{(\gamma,\delta)}(1) = \frac{\Gamma(n+\gamma+1)}{n!\Gamma(\gamma+1)}, \qquad P_n^{(\gamma,\delta)}(-1) = \frac{(-1)^n\Gamma(n+\delta+1)}{n!\Gamma(\delta+1)},$$

and for convenience to weight the ultraspherical polynomials so that

$$C_n^{(\gamma)}(x) = \frac{n!\Gamma(\gamma+1)}{\Gamma(n+\gamma+1)} P_n^{(\gamma-1/2,\gamma-1/2)}(x),$$

which gives $C_n^{(\gamma)}(1) = 1$ ($n = 0, 1, 2, \ldots$); this is not the usual standardization, but has the desirable properties that $C_n^{(0)}(x) = T_n(x)$, $C_n^{(1/2)}(x) = P_n(x)$, and $C_n^{(1)}(x) = (1/(n+1))U_n(x)$, where $T_n(x)$, $U_n(x)$ and $P_n(x)$ are Chebyshev polynomials of the first and second kinds and Legendre polynomials, respectively.

Now let $P_n^{(\gamma,\delta)}(ax+b)$ have the expansion

$$P_n^{(\gamma,\delta)}(ax+b) = \sum_{i=0}^{n} a_i(n) Y_i^{(\alpha)}(x), \tag{48}$$

where $P_n^{(\gamma,\delta)}(ax+b)$ satisfy the differential equation

$$[(1-(ax+b)^2)\mathrm{D}^2 + a[\delta-\gamma-(\mu+1)(ax+b)]\mathrm{D} + a^2 n(\mu+n)]P_n^{(\gamma,\delta)}(ax+b) = 0, \tag{49}$$

where $\mu = \gamma+\delta+1$, then the coefficients $a_i(n)$ satisfy the fourth-order recurrence relation

$$\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0,$$
$$i = n-1, \quad n-2, \ldots, 0, \tag{50}$$

where

$$\delta_{i0}=\frac{a^2(n-i)(i+n+\mu+1)(i+\alpha+1)_4}{4(2i+\alpha+1)_4},$$

$$\delta_{i1}=\frac{a(i+1)(i+\alpha+2)_3}{4(2i+\alpha+3)_2}\left[\left(\frac{1-b}{2}\right)(\mu+2i+1)+(i+\gamma+1)\right.$$
$$\left.+\frac{a[\alpha(2i+1)+2(3i+1)+4n(n+\mu)+(2-2i+\alpha)\mu]}{(2i+\alpha+2)(2i+\alpha+6)}\right],$$

$$\delta_{i2}=\frac{1}{4}(i+1)_2(\alpha+i+3)_2\left[\left(\frac{1-b^2}{4}\right)-\frac{a((\alpha-\mu+3)(1-b)-\alpha+2\gamma-2)}{(\alpha+2i+4)(\alpha+2i+6)}\right.$$
$$\left.+\frac{a^2[2i^2+2(\alpha+5)i+(3+\alpha)(3\mu-\alpha-1)+6n(n+\mu)]}{(\alpha+2i+3)_2(\alpha+2i+6)_2}\right],$$

$$\delta_{i3}=\frac{a(i+1)_3(i+\alpha+4)}{4(2i+\alpha+6)_2}\left[-\left(\frac{1-b}{2}\right)(2i+2\alpha-\mu+9)+(i+\alpha-\gamma+4)\right.$$
$$\left.+\frac{a(2(\alpha-\mu+3)i-(4+\alpha)(3\mu-2\alpha-7)-4n(n+\mu))}{(\alpha+2i+3)_2(\alpha+2i+6)_2}\right],$$

$$\delta_{i4}=-\frac{a^2(i+1)_4(i+\alpha+n+5)(i+\alpha-n-\mu+5)}{4(2i+\alpha+6)_4},$$

with $a_{n+s}(n)=0$, $s=1,2,3$ and $a_n(n)=\frac{a^n(n+\mu)_n}{n!(n+\alpha+1)_n}$. The solution of (50) is

$$a_i(n)=\frac{(\gamma+1)_n(-a)^i}{(i+\alpha+1)_i\,i!\,n!}\frac{(-n)_i(n+\mu)_i}{(\gamma+1)_i}\sum_{k=0}^{n-i}\frac{a^k}{k!}\frac{(-n+i)_k(n+\mu+i)_k}{(\gamma+i+1)_k(2i+\alpha+2)_k}$$
$$\times\,{}_2F_0\left[-k,-2i-k-\alpha-1;\frac{1-b}{2a}\right],\qquad i=0,1,\ldots,n. \tag{51}$$

Corollary 4. *In the connection problem*

$$P_n^{(\gamma,\delta)}(x)=\sum_{i=0}^{n}a_i(n)Y_i^{(\alpha)}(x), \tag{52}$$

the coefficients $a_i(n)$ are given by

$$a_i(n)=\frac{(n+\mu)_i(\gamma+1)_n}{i!(n-i)!(\gamma+1)_i(i+\alpha+1)_i}\sum_{k=0}^{n-i}\frac{1}{k!}\frac{(-n+i)_k(n+\mu+i)_k}{(\gamma+i+1)_k(2i+\alpha+2)_k}$$
$$\times\,{}_2F_0\left[-k,-k-2i-\alpha-1;\frac{1}{2}\right],\qquad i=0,1,\ldots,n. \tag{53}$$

Corollary 5. *The link between ultraspherical–Bessel connection problem is given by*

$$C_n^{(\nu)}(x)=\sum_{i=0}^{n}a_i(n)Y_i^{(\alpha)}(x), \tag{54}$$

where the coefficients $a_i(n)$ are given by

$$a_i(n)=\frac{n!(n+2\nu)_i}{i!(n-i)!(\nu+1/2)_i(i+\alpha+1)_i}\sum_{k=0}^{n-i}\frac{1}{k!}\frac{(-n+i)_k(n+2\nu+i)_k}{(\nu+i+1/2)_k(2i+\alpha+2)_k}$$
$$\times\,{}_2F_0\left[\begin{matrix}-k,-k-2i-\alpha-1\\ -\end{matrix};\frac{1}{2}\right],\qquad i=0,1,\ldots,n. \tag{55}$$

Remark 4. It is worth noting that all the connection problems between the three orthogonal polynomials, Chebyshev polynomials of the first and second kinds and Legendre polynomials and Bessel polynomials can be easily deduced by taking $\nu = 0, 1, 1/2$ in relations (54) and (55), respectively.

6.3. The Laguerre–Bessel connection problem

In this problem

$$L_n^{(\gamma)}(ax+b) = \sum_{i=0}^{n} a_i(n) Y_i^{(\alpha)}(x), \tag{56}$$

where $L_n^{(\gamma)}(ax+b)$ are Laguerre polynomials, which satisfy the differential equation

$$[(ax+b)\mathrm{D}^2 + a(1+\gamma-(ax+b))\mathrm{D} + a^2 n]L_n^{(\gamma)}(ax+b) = 0. \tag{57}$$

The coefficients $a_i(n)$ satisfy the recurrence relation

$$\begin{aligned}&\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0,\\ &\qquad i = n-1, \quad n-2, \ldots, 0,\end{aligned} \tag{58}$$

where

$$\begin{aligned}
\delta_{i0} &= \frac{a^2(n-i)(i+\alpha+1)_4}{(2i+\alpha+1)_4},\\
\delta_{i1} &= -\frac{a(i+1)(i+\alpha+2)_3}{2(2i+\alpha+3)_2}\left[(b-i-\gamma-1) + \frac{2a(2i+4n-2-\alpha)}{(2i+\alpha+2)(2i+\alpha+6)}\right],\\
\delta_{i2} &= \frac{1}{4}(i+1)_2(\alpha+i+3)_2\left[b - \frac{2a(2b+\alpha-2\gamma+2)}{(\alpha+2i+4)(\alpha+2i+6)} + \frac{12a^2(\alpha+2n+3)}{(\alpha+2i+3)_2(\alpha+2i+6)_2}\right],\\
\delta_{i3} &= \frac{a(i+1)_3(i+\alpha+4)}{2(2i+\alpha+6)_2}\left[-(i+\alpha+b-\gamma+4) + \frac{2a(2i+3\alpha+4n+12)}{(\alpha+2i+4)(\alpha+2i+8)}\right],\\
\delta_{i4} &= \frac{a^2(i+1)_4(i+\alpha+n+5)}{(2i+\alpha+6)_4},
\end{aligned}$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3$ and $a_n(n) = \frac{a^n(-2)^n}{n!(n+\alpha+1)_n}$. The solution of (58) is

$$\begin{aligned}a_i(n) = {}& \frac{(2a)^i}{(i+\alpha+1)_i\, i!}\frac{(-n)_i(\gamma+1)_n}{n!(\gamma+1)_i}\sum_{k=0}^{n-i}\frac{(-2a)^k}{k!(2i+\alpha+2)_k}\frac{(-(n-i))_k}{(\gamma+i+1)_k}\\ &\times {}_2F_0\left[\begin{matrix}-k, -2i-k-\alpha-1\\ -\end{matrix}; -\frac{b}{2a}\right], \qquad i = 0, 1, \ldots, n.\end{aligned} \tag{59}$$

Corollary 6. *In the connection problem*

$$L_n^{(\gamma)}(ax) = \sum_{i=0}^{n} a_i(n) Y_i^{(\alpha)}(x),$$

the coefficients $a_i(n)$ are given by

$$\begin{aligned}a_i(n) = {}& \frac{(2a)^i}{(i+\alpha+1)_i\, i!}\frac{(-n)_i(\gamma+1)_n}{n!(\gamma+1)_i}\, {}_1F_2\left[\begin{matrix}-(n-i)\\ 2i+\alpha+2, \gamma+i+1\end{matrix}; -2a\right],\\ & i = 0, 1, \ldots, n.\end{aligned} \tag{60}$$

6.4. The Hermite–Bessel connection problem

In this problem

$$H_n(ax+b) = \sum_{i=0}^{n} a_i(n) Y_i^{(\alpha)}(x),$$

where $H_n(x)$ are Hermite polynomials, which satisfy the differential equation

$$[\mathrm{D}^2 - 2a(ax+b)\mathrm{D} + 2a^2 n] H_n(ax+b) = 0. \tag{61}$$

The coefficients $a_i(n)$ satisfy the recurrence relation

$$\delta_{i0} a_i(n) + \delta_{i1} a_{i+1}(n) + \delta_{i2} a_{i+2}(n) + \delta_{i3} a_{i+3}(n) + \delta_{i4} a_{i+4}(n) = 0, \quad i = n-1, \quad n-2, \ldots, 0, \tag{62}$$

where

$$\delta_{i0} = \frac{2a^2(i+\alpha+1)_4(n-i)}{(2i+\alpha+1)_5},$$

$$\delta_{i1} = -a(i+1)(i+\alpha+2)_3 \left[\frac{b}{(2i+\alpha+3)_3} - \frac{2a(4n-2i+\alpha+2)}{(2i+\alpha+2)_5} \right],$$

$$\delta_{i2} = \frac{1}{4}(i+1)_2(i+\alpha+3)_2 \left[\frac{1}{(2i+\alpha+5)} - \frac{8ab}{(2i+\alpha+4)_3} + \frac{24a^2(2n+\alpha+3)}{(2i+\alpha+3)_5} \right],$$

$$\delta_{i3} = a(i+\alpha+4)(i+1)_3 \left[-\frac{b}{(2i+\alpha+5)_3} + \frac{2a(4n+2i+3\alpha+12)}{(2i+\alpha+4)_5} \right],$$

$$\delta_{i4} = \frac{2a^2(i+1)_4(n+i+\alpha+5)}{(2i+\alpha+5)_5},$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3$ and $a_n(n) = \frac{a^n 2^{2n}}{(n+\alpha+1)_n}$. The solution of (62) is

$$a_i(n) = \frac{(2a)^i n!}{(i+\alpha+1)_i i!} \sum_{k=0}^{[\frac{n-i}{2}]} \frac{(-1)^k 2^{n-2k} b^{n-2k-i}}{k!(n-2k-i)!} \, {}_1F_1 \left[\begin{matrix} -(n-2k-i) \\ 2i+\alpha+2 \end{matrix} ; \frac{2a}{b} \right], \quad i = 0, 1, \ldots, n. \tag{63}$$

Corollary 7. *In the connection problem*

$$H_n(ax) = \sum_{i=0}^{n} a_i(n) Y_i^{(\alpha)}(x),$$

the coefficients $a_i(n)$ are given by

$$a_i(n) = \frac{(-1)^{n-i} n!}{(i+\alpha+1)_i i!} \sum_{k=0}^{[\frac{n-i}{2}]} \frac{(-1)^k}{k!(n-2k-i)!} \frac{(4a)^{n-2k}}{(2i+\alpha+2)_{n-2k-i}}, \qquad i = 0, 1, \ldots, n. \tag{64}$$

Remark 5. The expansions and connection coefficients in series of ordinary Bessel polynomials $y_n(x)$ can be obtained directly from those of the generalized Bessel polynomials $Y_n^{(\alpha)}(x)$, by taking $\alpha = 0$.

Remark 6. It is to be noted that one of our goals is to emphasize the systematic character and simplicity of our algorithm, which allows one to implement it in any computer algebra (here the Mathematica (1999)) symbolic language used.

To end this paper, we wish to report that this work deals with formulae associated with the Bessel coefficients for the moments of a general-order derivative of differentiable functions and with the connection coefficients between Bessel–Bessel, Jacobi–Bessel, Laguerre–Bessel and Hermite–Bessel and other combinations with different parameters. These formulae can be used to facilitate greatly the setting up of the algebraic systems to be obtained by applying the spectral methods for solving differential equations with polynomial coefficients of any order.

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References

Abramowitz M and Stegun I A (ed) 1970 *Handbook of Mathematical Functions* (*Appl. Maths Series* vol 55) (New York: National Bureau of Standards)

Agarwal R P 1954 On Bessel polynomials *Can. J. Math.* **6** 410–5

Àlvarez-Nodarse R and Ronveaux A 1996 Recurrence relations for connection coefficients between q-orthogonal polynomials of discrete variables in the non uniform lattice $x(s) = q^{2s}$ *J. Phys. A: Math. Gen.* **29** 7165–75

Al-Salam W 1957 The Bessel polynomials *Duke Math. J.* **24** 529–45

Àlvarez-Nodarse, Yàñez R J and Dehesa J S 1998 Modified Clebsch–Gordan-type expansions for products of discrete hypergeometric polynomials *J. Comput. Appl. Math.* **89** 171–97

Andrews G E, Askey R and Roy R 1999 *Special Functions* (Cambridge: Cambridge University Press)

Area I, Godoy E, Ronveaux A and Zarzo A 1998 Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: discrete case *J. Comput. Appl. Math.* **89** 309–25

Askey R and Gasper G 1971 Jacobi polynomials expansions of Jacobi polynomials with non-negative coefficients *Proc. Camb. Phil. Soc.* **70** 243–55

Ben-Yu G 1998 *Spectral Methods and Their Applications* (London: World Scientific)

Cannon B 1937 On the convergence of series of polynomials *Proc. London Math. Soc.* **43** 348–64

Canuto C, Hussaini M Y, Quarteroni A and Zang T A 1988 *Spectral Methods in Fluid Dynamics* (Berlin: Springer)

Carlitz L 1957 On the Bessel polynomials *Duke Math. J.* **24** 151–62

Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

Coutsias E A, Hagstorm T and Torres D 1996 An efficient spectral methods for ordinary differential equations with rational function coefficients *Math. Comput.* **65** 611–35

Doha E H 1990 An accurate solution of parabolic equations by expansion in ultraspherical polynomials *J. Comput. Math. Appl.* **19** 75–88

Doha E H 1991 The coefficients of differentiated expansions and derivatives of ultraspherical polynomials *J. Comput. Math. Appl.* **21** 115–22

Doha E H 1994 The first and second kinds Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function *Int. J. Comput. Math.* **51** 21–35

Doha E H 1998 The ultraspherical coefficients of the moments of a general-order derivative of an infinitely differentiable function *J. Comput. Appl. Math.* **89** 53–72

Doha E H 2000 The coefficients of differentiated expansions of double and triple ultraspherical polynomials *Ann. Univ. Sci. Budapest, Sect. Comput.* **19** 57–73

Doha E H 2002a On the coefficients of differentiated expansions and derivatives of Jacobi polynomials *J. Phys. A: Math. Gen.* **35** 3467–78

Doha E H 2002b On the coefficients of integrated expansions and integrals of ultraspherical polynomials and their applications for solving differential equations *J. Comput. Appl. Math.* **139** 275–98

Doha E H 2003a Explicit formulae for the coefficients of integrated expansions for Jacobi polynomials and their integrals *Integral Transforms Spec. Funct.* **14** 69–86

Doha E H 2003b On the connection coefficients and recurrence relations arising from expansions in series of Laguerre polynomials *J. Phys. A: Math. Gen.* **36** 5449–62

Doha E H 2004a On the connection coefficients and recurrence relations arising from expansions in series of Hermite polynomials *Integral Transforms Spec. Funct.* **15** 13–29

Doha E H 2004b On the construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials *J. Phys. A: Math. Gen.* **37** 657–75

Doha E H and Abd-Elhameed W M 2002 Efficient spectral Galerkin-algorithms for direct solution of second-order equations using ultraspherical polynomials *SIAM J. Sci. Comput.* **24** 548–71

Doha E H and Al-Kholi F M R 2001 An efficient double Legendre spectral method for parabolic and elliptic partial differential equations *Int. J. Comput. Math.* **78** 413–32

Doha E H and El-Soubhy S I 1995 On the Legendre coefficients of the moments of general-order derivative of an infinitely differentiable function *Int. J. Comput. Math.* **56** 107–22

Doha E H and Helal M A 1997 An accurate double Chebyshev spectral approximation for parabolic partial differential equations *J. Egypt. Math. Soc.* **5** 83–101

Evans W D, Everitt W N, Kwon K H and Littlejohn L L 1993 Real orthogonalizing weights for Bessel polynomials *J. Comput. Appl. Math.* **49** 51–7

Gasper G 1974 Project formulas for orthogonal polynomials of a discrete variable *J. Math. Anal. Appl.* **45** 176–98

Godoy E, Ronveaux A, Zarzo A and Area I 1997 Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case *J. Comput. Appl. Math.* **84** 257–75

Grosswald E 1978 Bessel polynomials *Lecture Notes Math.* **698** 1–182

Haidvogel D B and Zang T 1979 The accurate solution of Poisson's equation by Chebyshev polynomials *J. Comput. Phys.* **30** 167–80

Han S S and Kwon K H 1991 Spectral analysis of Bessel polynomials in Krein space *Quaest. Math.* **14** 327–35

Helal M A 2001 A Chebyshev spectral method for solving Korteweg–de Vries equation with hydrodynamical application *Chaos Solitons Fractals* **12** 943–50

Karageorghis A 1988a A note on the Chebyshev coefficients of the general-order derivative of an infinitely differentiable function *J. Comput. Appl. Math.* **21** 129–32

Karageorghis A 1988b A note on the Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function *J. Comput. Appl. Math.* **21** 383–6

Karageorghis A and Phillips T N 1989 On the coefficients of differentiated expansions of ultraspherical polynomials *ICASE Report no 89-65* NASA Langley Research Center, Hampton, VA

Karageorghis A and Phillips T N 1992 *Appl. Numer. Math.* **9** 133–41

Koepf W and Schmersau D 1998 Representations of orthogonal polynomials *J. Comput. Appl. Math.* **90** 57–94

Krall H L and Frink O 1949 A new class of orthogonal polynomials : the Bessel polynomials *Trans. Am. Math. Soc.* **65** 100–5

Lewanowicz S 2002 Recurrences of coefficients of series expansions with respect to classical orthogonal polynomials *Appl. Math.* **29** 97–116

Lewanowicz S, Godoy E, Area I, Ronveaux A and Zarzo A 2000 Recurrence relations for coefficients of Fourier series expansions with respect to q-classical orthogonal polynomials *Numer. Algorithms* **23** 31–50

Lewanowicz S and Woźny P 2001 Algorithms for construction of recurrence relations for the coefficients of expansions in series of classical orthogonal polynomials *Preprint* Institute of Computer Science University of Wroclaw online at http://www.ii.uni.wroc.pl/sle/publ.html

Luke Y L 1969 *The Special Functions and Their Approximations* vols 1 and 2 (New York: Academic)

Nassif M 1954 Note on the Bessel polynomials *Trans. Am. Math. Soc.* **77** 408–12

Phillips T N 1988 On the Legendre coefficients of a general-order derivative of an infinitely differentiable function *IMA. J. Numer. Anal.* **8** 455–9

Rainville E D 1960 *Special Functions* (New York: Macmillan)

Ronveaux A, Belmehdi S, Godoy E and Zarzo A 1996 Recurrence relation approach for connection coefficients. Applications to classical discrete orthogonal polynomials *CRM Proc. and Lecture notes* vol 9 (Centre de Researches Mathématiques) pp 319–35

Ronveaux A, Zarzo A and Godoy E 1995 Recurrence relations for connection coefficients between two families of orthogonal polynomials *J. Comput. Appl. Math.* **62** 67–73

Sánchez-Ruiz J and Dehesa J S 1998 Expansions in series of orthogonal polynomials *J. Comput. Appl. Math.* **89** 155–70

Siyyam H I and Syam M I 1997 An accurate solution of Poisson equation by the Chebyshev–Tau methods *J. Comput. Appl. Math.* **85** 1–10

Szegö G 1985 *Orthogonal Polynomials (American Mathematical Society Colloquium Publications* vol 23*)* (Providence, RI: American Mathematical Society)

Voigt R G, Gottlieb D and Hussaini M Y 1984 *Spectral Methods for Partial Differential Equations* (Philadelphia, PA: SIAM)

Whittaker J M 1949 *Sur les séries de base de polynomes quelconques* (Paris: Gauthier-Villars)

Wolfram Research Inc. 1999 *Mathematica Version 4.0 (Wolfram Research Champaign)*

Zarzo A, Area I, Godoy E and Ronveaux A 1997 Results for some inversion problems for classical continuous and discrete orthogonal polynomials *J. Phys. A: Math. Gen.* **30** 35–40